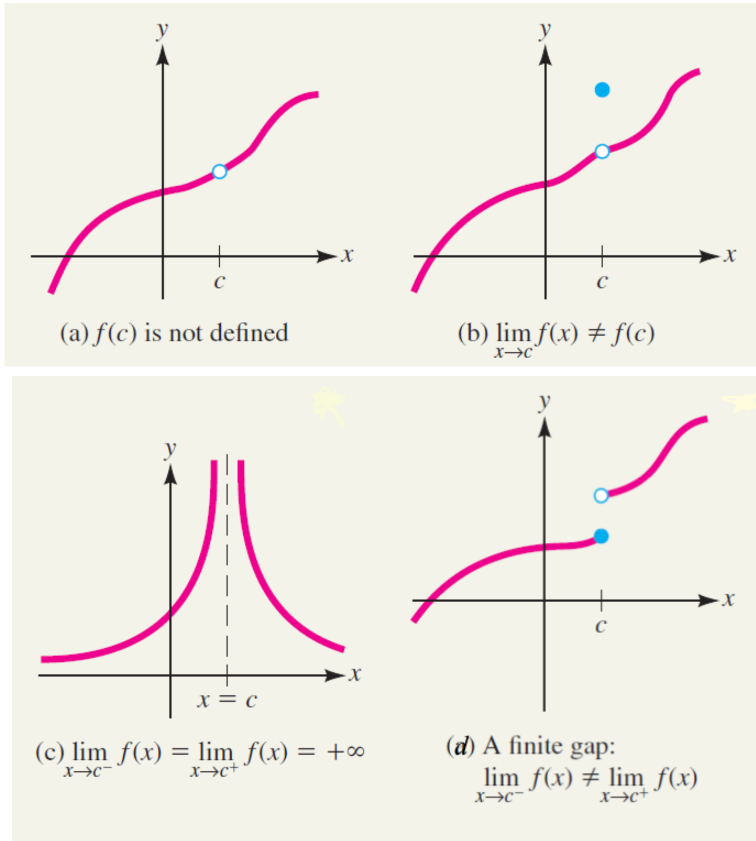



(a) $f(c)$ is not defined

(b) $\lim_{x\to c} f(x) \neq f(c)$

(c) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = +\infty$

(d) A finite gap: $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

**Example 3.1.9.** For what value of $A$ such that the following function is continuous at all $x$?

$$f(x) = \begin{cases} x^2 + x - 1 & \text{if } x \leq 0, \\ x + \underbrace{A}_{\text{constant}} & \text{if } x > 0. \end{cases}$$

← continuous on $(0, \infty)$

← continuous on $(-\infty, 0)$

*Solution.* Because $x^2 + x - 1$ and $x + A$ are polynomials, they are continuous everywhere except possibly at $x = 0$. Also $f(0) = 0^2 + 0 - 1 = -1$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2 + x - 1) = -1$$

" $0 + 0 - 1$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x + A) = A.$$

" $0 + A$

For $\lim_{x\to 0} f(x)$ to exist, the left hand limit and the right hand limit must be equal. So we must have $A = -1$. In which case

$$\lim_{x\to 0} f(x) = -1 = f(0).$$

want $f(x)$ to be continuous at 0.

This means that $f(x)$ is continuous for all $x$ only when $A = -1$. ■

**Proposition 3.1.2.** $f(x)$ is continuous at $x = c$ if and only if

$$\lim_{h \to 0} f(c+h) = f(c).$$

*Proof.* Let $h = x - c$. Then $h \to 0$ as $x \to c$.

distant from $x$ to $c$

$$\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h).$$

□

view as a composition function. $u = x^3+1$

$f(x) = \sqrt[3]{u}$

*Exercise* 3.1.1.

$f(x)$ ||

1. Show that $\sqrt[3]{x^3+1}$ is a continuous function.
2. Show that $\left|\dfrac{x+1}{x-1}\right|$ is a continuous function on $\mathbb{R}\backslash\{1\}$.
3. Let
$$f(x) = \begin{cases} x^2 - 1, & x \le 0, \\ x + a, & x > 0. \end{cases}$$
Find $a$ such that $f(x)$ is continuous at 0. (Ans: $a = -1$)

**Example 3.1.10** (Using continuity to compute limits). $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = ?$

change variable, let $u = \frac{1}{x}$

## 3.2 Continuity on $[a, b]$

**Definition 3.2.1.** Let $f : (a, b) \to \mathbb{R}$ be a function. Then $f$ is said to be continuous on $(a, b)$ if it is continuous at every point on $(a, b)$.

Next, let's assume $f : [a, b] \to \mathbb{R}$ be a function. What's the meaning of $f$ being continuous at one of the end point $a$? $\lim_{x\to a} f(x)$ does not make sense because $f$ is not defined on $x < a$. So to define the continuity at $x = a$, we only concern about the value $x > a$. Similarly, to discuss about the continuity at $x = b$, we only concern about the value $x < b$.

$a$ $b$

$f$ is defined

**Definition 3.2.2.** Let $f : [a, b] \to \mathbb{R}$ be a function.
Then $f$ is said to be continuous at $a$ if

$$\lim_{x\to a^+} f(x) = f(a).$$

$f$ is said to be continuous at $b$ if

$$\lim_{x\to b^-} f(x) = f(b).$$

Then $f$ is said to be a continuous function on $[a, b]$ if $f$ is continuous on $a \le x \le b$.

**Example 3.2.1.** Discuss the continuity of the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x-1}{x} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

continuous except $x = 0$

use definition of continuity for end points

*Solution.* $f(x)$ is continuous on $(0, 1)$. $f(x)$ is also continuous at $x = 1$, but $\lim_{x \to 0^+} f(x)$ does not exists. So $f$ is not continuous at $x = 0$. ■

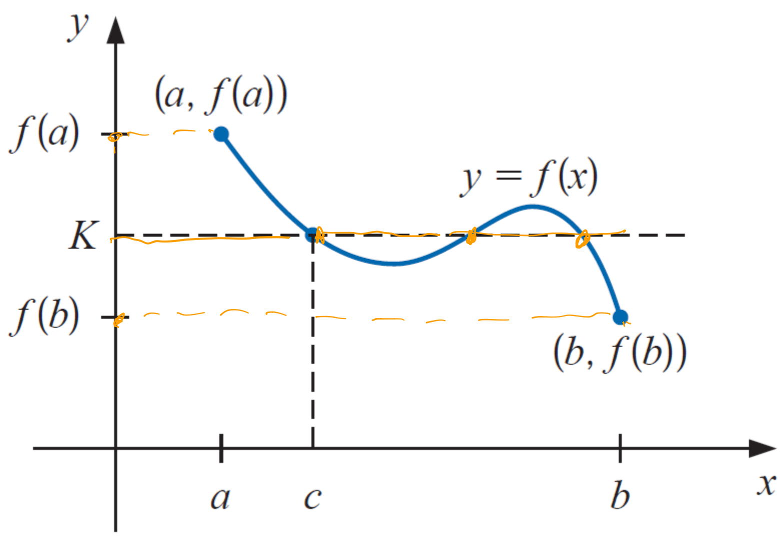
$\lim_{x \to 0^+} \frac{x-1}{x}$ " $=$ $\frac{-1}{0^+}$ " $= -\infty \neq f(0)$

finite & closed interval?

**Theorem 3.2.1** (Intermediate Value Theorem or Intermediate Value Property)**.** *Suppose $f$ is a continuous function on $[a, b]$ and $K$ is a number between $f(a)$ and $f(b)$. Then there exist a number $c$, between $a$ and $b$, such that $f(c) = K$.*

but $c$ might not be unique, and the thm doesn't tell us where $c$ is

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the $y$-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.




### *Application: Root Finding*

If $f(x)$ is continuous on $[a, b]$, $f(a)$ and $f(b)$ change sign, then, there exists at least one root of the function, that is, exists at least one $c \in (a, b)$, such that $f(c) = 0$.

$f(x) = 0$ has a solution (i.e. $f(x)$ has a root)

apply the intermediate thm with $K = 0$ (btwn $f(a)$, $f(b)$ which have different signs)

**Example 3.2.2.** Show that $f(x) = x^5 - x + 1$ has a root.

take $x$ to be very large ($x \gg 0$)
$x^5$ dominates the lower order terms
so $f(x)$ is positive for all sufficiently large $x$. $\lim_{x \to \infty} f(x) = \infty$

take $x$ ... very negative $(x \ll 0)$

$\lim_{x \to -\infty} f(x) = -\infty$

so $f(x)$ is negative for all sufficiently negative $x$.

$\rightsquigarrow$ $f(x)$ has a root

*Solution.* Aim: find $a$, $b$, such that $f(a)$, $f(b)$ change sign. Since

$$f(-2) = -29, \quad f(0) = 1,$$

and $f$ is continuous on $[-2, 0]$. By Intermediate value theorem, there exists $c \in (-2, 0)$, such that $f(c) = 0$.

■

*Remark.* Although we don't know how to find the root, we know a root exists.

**Example 3.2.3.** 1. All odd functions have a root. $f(x) = -f(-x)$

2. All polynomials of odd degrees have a root.

*Exercise* 3.2.1. Show $2^x = \dfrac{1}{x^2}$ has a solution.

rewrite as $\underbrace{2^x - \frac{1}{x^2}}_{f(x)} = 0$

$\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = -\infty$

**Theorem 3.2.2** (Extreme Value Theorem). *If $f(x)$ is continuous on $[a, b]$, then $f$ must attain an absolute maximum and absolute minimum, that is, there exist $c$, $d$ in $[a, b]$ such that*

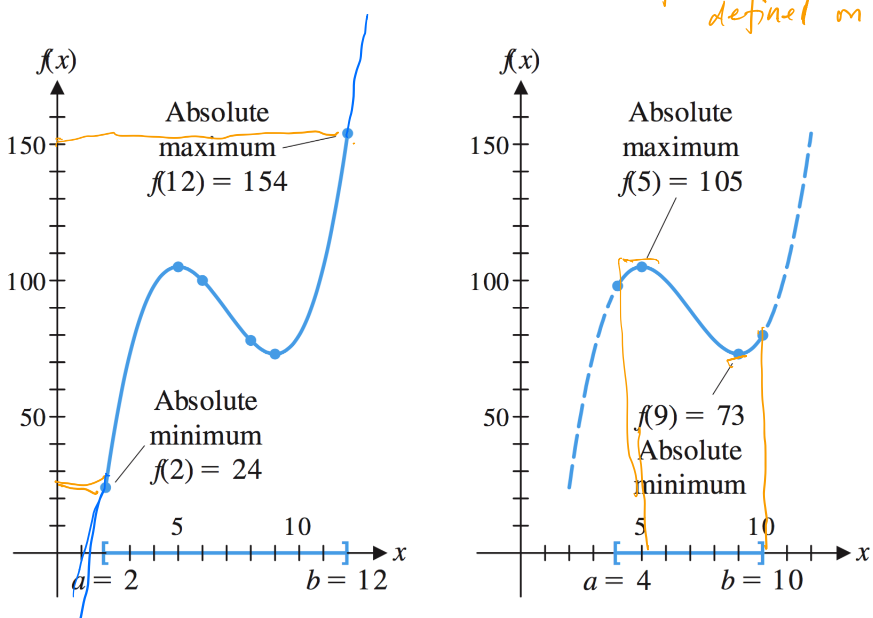
$$f(c) \le f(x) \le f(d),$$

*for all $x \in [a, b]$.*

closed, finite interval !!

**Example 3.2.4.** Absolute extreme for $f(x) = x^3 - 21x^2 + 135x - 170$ for various closed intervals.

defined on $\mathbb{R}$




(A) $[a, b] = [2, 12]$ (B) $[a, b] = [4, 10]$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

$f$ has neither an absolute maximum nor a absolute minimum

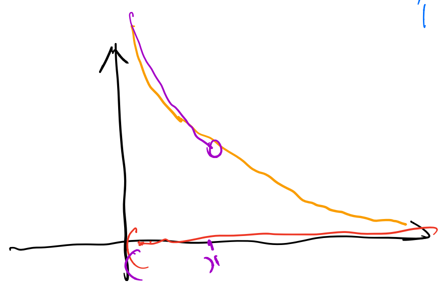
*Exercise* 3.2.2 (Hard!). Derive the extreme value theorem from the intermediate value theorem.

*Remark*. Caveat: The extreme value theorem and the intermediate value thm only works on *finite* and closed intervals! E.g. Consider the previous example on $\mathbb{R}$ or $\frac{1}{x}$ on $\mathbb{R}^+$.

E.g. $f(x) = \frac{1}{x}$ on $(0, \infty)$

infinite open interval

has neither absolute max nor absolute min on $(0, \infty)$



E.g. $f(x) = \frac{1}{x}$, on $(0, 1)$

has neither abs. min nor abs. max

**Question**: How to find the absolute maximum and minimum?

Ans: (for "good" functions) Differentiation!

Feb 2

**Ex.** Compute $\lim_{x \to \infty} \sin\left(\frac{1}{x}\right)$

**Sol:** change of variable: $u = \frac{1}{x}$ when $x \to \infty$ $u \to 0^+$

$$\lim_{x \to \infty} \sin\left(\frac{1}{x}\right) = \lim_{u \to 0^+} \sin u = \sin 0 = 0 \qquad \square$$

use the continuity of sin. and the composition rule for continuity.

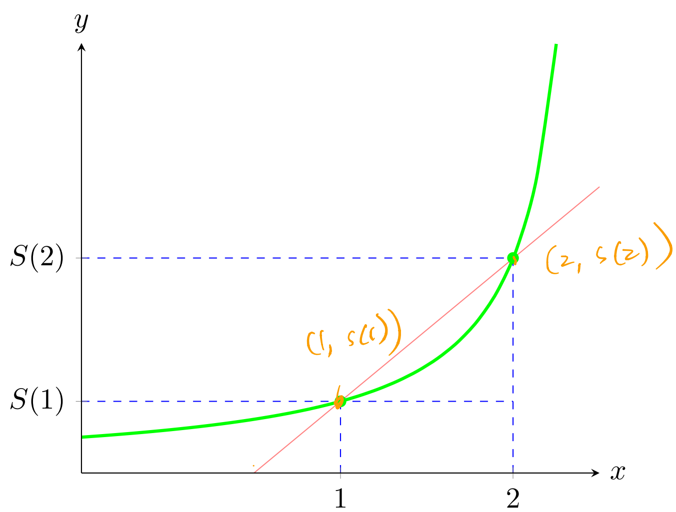
**MATH1520 University Mathematics for Applications** **Spring 2021**

# Chapter 4: Differentiation I

**Learning Objectives**:
(1) Define the derivatives, and study its basic properties.
(2) Study the relationship between differentiability and continuity.
(3) Use the constant multiple rule, sum rule, power rule, product rule, quotient rule and chain rule to find derivatives.
(4) Explore logarithmic differentiation.

## 4.1 Motivation & Definition

**Motivation from physics: velocity** Suppose an object is moving along $x$-axis from the origin to right. Let $S = S(t)$ be the position of the object at time $t$. What is the average velocity of this object from $t = 1$ to $t = 2$?



in the value of position function

$$\text{Average velocity from } t=1 \text{ to } t=2 = \frac{\text{Change } \cancel{\text{of distance}}}{\text{Change of time}} = \frac{\Delta S}{\Delta t} = \frac{S(2)-S(1)}{2-1} = \text{slope of secant line passing through } (1, S(1)) \text{ and } (2, S(2))$$

difference quotient

**Question**: What is the instantaneous velocity at $t = 1$? ( i.e. taking $\Delta t \to 0$ in the difference quotient )

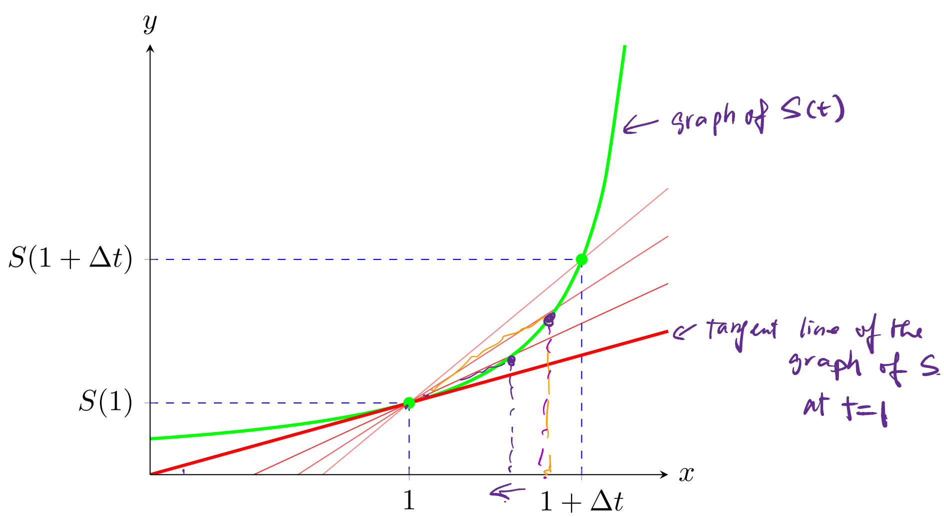
Idea: Average velocity from $t = 1$ to $t = 1 + \Delta t$ is $\dfrac{S(1+\Delta t) - S(1)}{\Delta t}$, where $\Delta t$ is small.

Let $\Delta t \to 0$, the instantaneous velocity at $t = 1$ is defined to be

$$S'(1) = \lim_{\Delta t \to 0} \frac{S(1+\Delta t) - S(1)}{\Delta t},$$

slope of the tangent line through $t=1$

which is called the **derivative** of $S$ at $t = 1$. $S'(1)$ describes the rate of change of $S(t)$ at $t = 1$.




*Remark*. *Terminology*: The term “velocity” takes the direction of motion into account; it can be positive or negative. The term “speed” only takes into account the rate of change, disregarding the direction. It is the absolute value of the velocity.

**Definition 4.1.1.** The **derivative** of $f(x)$ is the function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{4.1}$$

The process of computing the derivative is called **differentiation**, and we say that $f(x)$ is **differentiable** at $x = x_0$ if $f'(x_0)$ exists; that is, $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists.

*Remark.* 1. By definition, if $f(x_0)$ is not well-defined, we cannot define $f'(x_0)$. So $f(x)$ must not be differentiable at $x = x_0$.

2. Another equivalent formula:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

3. $$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

   is called **difference quotient.**

4. $f'(x_0)$ describes the rate of change of $f(x)$ at $x = x_0$.

5. When we say that we use **the first principle** to find derivatives, we mean that we use the definition (4.1) to find the derivative. However, later we will learn faster techniques to find derivatives.

**Geometrical interpretation of differentiation:** $f'(x_0)$ is the slope of tangent line to the curve of $f(x)$ at $x = x_0$.

**Example 4.1.1.** Let $f(x) = x^2$. Then (i) prove that $f(x)$ is differentiable at $x = 1$; (ii) find $f'(1)$ and the equation of the tangent line to the ~~curve~~ graph of $f(x)$ at $x = 1$.

*Solution.* (i) By the definition, at $x = 1$

$1 + 2\Delta x + (\Delta x)^2$

$$f'(1) := \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(1 + \Delta x)^2 - \cancel{1^2}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2 + \Delta x)$$
$$= 2,$$

So, $f$ is differentiable at $1$, and $f'(1) = 2$.

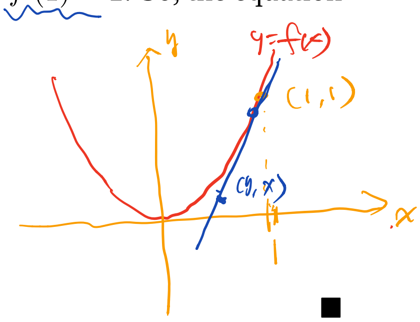
(ii) The tangent line passes through $(1, f(1)) = (1, 1)$ with slope $f'(1) = 2$. So, the equation of the tangent line is

Thus

$$\text{slope of the tangent line} = \frac{y - f(1)}{x - (1)} = 2.$$

$$y - 1 = 2(x - 1)$$

$$y = 2x - 1.$$




■

**Definition 4.1.2.** If $f(x) : A \to \mathbb{R}$ is differentiable at every point $x \in A$, then $f(x)$ is said to be a differentiable function in $A$, and the derivative function $f'(x) : A \to \mathbb{R}$ is well-defined.

**Example 4.1.2.** Let $f(x) = x^2$. Prove that $f(x)$ is differentiable on $\mathbb{R}$, and find $f'(x)$.

*Solution.* For any $x \in \mathbb{R}$,

$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ $\qquad$ $(x+\Delta x - x)(x+\Delta x + x)$

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}(2x+\Delta x) = 2x.$$

So, $f$ is differentiable at $x$, and $f'(x) = 2x$. ∎

for all real $x$ $\qquad$ so $f$ is a differentiable function on $\mathbb{R}$

**Notation**: For $y = f(x) = x^2$,

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = 2x; \quad f'(4) = \left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{df}{dx}\right|_{x=4} = 2 \cdot 4 = 8.$$

"infinitesimal $\Delta x$"

*Question* Where does the minimum of $x^2$ occur? (Hint: what is the slope of the tangent line at the minimum?)

**Example 4.1.3.** Let $f(x) = \frac{x+1}{x-1}$. Using the definition of derivatives, compute $f'(x)$ for $x \neq 1$.

*Solution.*

$$\Delta f = f(x+\Delta x) - f(x) = \frac{x+\Delta x+1}{x+\Delta x-1} - \frac{x+1}{x-1} \cdot \frac{(x+\Delta x-1)}{(x+\Delta x-1)}$$
$$= \frac{(x-1)(x+\Delta x+1)-(x+1)(x+\Delta x-1)}{(x-1)(x+\Delta x-1)}$$
$$= \frac{-2\Delta x}{(x-1)(x+\Delta x-1)}.$$

Therefore

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{-2}{(x-1)(x+\Delta x-1)}\right)$$
$$= \frac{\lim_{\Delta x\to 0}(-2)}{\lim_{\Delta x\to 0}(x-1)(x+\Delta x-1)} = \frac{-2}{(x-1)^2}.$$

when $x \neq 1$

$\frac{-2}{(x-1)(x-1)}$

∎

**Example 4.1.4.** Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

*Solution.*

use the formula $a^2-b^2 = (a+b)(a-b)$

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \Leftarrow (x+\Delta x)-x^2 \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}.\end{aligned}$$

So, $\left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}}, x > 0.$ ■

**Example 4.1.5.** Find the derivative of $f(x) = \sqrt[3]{x}$.
**Hint**: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

*Solution.* For any $x \neq 0$,

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt[3]{x+\Delta x} - \sqrt[3]{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(\sqrt[3]{x+\Delta x} - \sqrt[3]{x})((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)}{\Delta x((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)} \\ &= \lim_{h \to 0} \frac{x + \Delta x - x}{\Delta x((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)} \\ &= \lim_{\Delta x \to 0} \frac{1}{(\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2} \\ &= \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3}x^{-\frac{2}{3}}.\end{aligned}$$

For $x = 0$,

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x} - \sqrt[3]{0}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{\frac{2}{3}}} \quad \text{does not exist.}$$

So,

$$(x^{1/3})' = \begin{cases} \frac{1}{3}x^{-\frac{2}{3}}, & x \neq 0 \\ \text{Not exist at } x = 0, \text{ i.e. } x^{\frac{1}{3}} \text{ not differentiable at } 0 \end{cases}$$

■

**Example 4.1.6.** Discuss the differentiability of $f(x) = |x|$.

" $\begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$ this is continuous function

( Rmk: A differentiable function is continuous but not vice versa )

*Solution.* For $x_0 > 0$,

$$\frac{dx}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x) - x_0}{\Delta x} = 1.$$

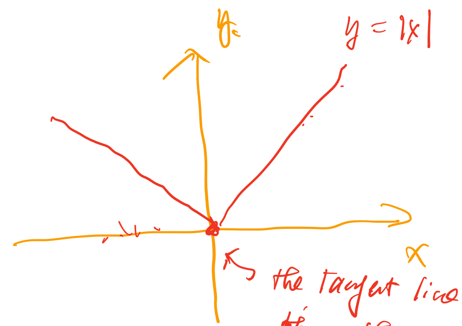
For $x_0 < 0$,

$$\frac{d(-x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} == \lim_{\Delta x \to 0} \frac{-(x_0 + \Delta x) - (-x_0)}{\Delta x} = -1.$$

For $x_0 = 0$.

$$\lim_{\Delta x \to 0+} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0+} \frac{\Delta x}{\Delta x} = 1.$$

$$\lim_{\Delta x \to 0-} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0-} \frac{-\Delta x}{\Delta x} = -1.$$




the tangent line thru this point does not exist

$1 \neq -1$, so $f$ is not differentiable at $x = 0$. So,

$\lim_{\Delta x \to 0} \frac{\partial f}{\partial x}\Big|_{x=0}$ doesn't exist

$$(|x|)' = \begin{cases} 1 & \text{if } x > 0, \\ \text{undefined} & \text{if } x = 0. \\ -1 & \text{if } x < 0, \end{cases}$$

■

## 4.2 Properties of derivatives

### 4.2.1 Differentiation and Continuity

**Proposition 1.** $f(x)$ is differentiable at $x = x_0 \implies f(x)$ is continuous at $x = x_0$.

*Proof.* Suppose $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists, then

$$\begin{aligned} \lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0. \end{aligned}$$

So, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (f(x) - f(x_0)) + \lim_{x \to x_0} f(x_0) = 0 + f(x_0) = f(x_0)$, that is, $f(x)$ is continuous at $x_0$. □